A BINOMIAL THEOREM, EXPRESSED IN FORM OF A FACTORIAL, WHICH IS *ALWAYS* CONVERGENT.*

By Chas. H. Kummell, Washington, D. C.

The binomial series:

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n}{1}\cdot\frac{n-1}{2}x^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^3 + \dots$$

$$+ \frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-m+1}{m}x^m + \dots$$

$$= 1 + \frac{n}{1}x\,(1 + \frac{n-1}{2}x\,(1 + \frac{n-2}{3}\,(1 + \dots \tag{1}$$

is true for any value of x only if n is a positive integer. In all other cases we must have

$$1 > x > -1.$$

About six years ago Dr. Martin communicated to the Mathematical Section of the Philosophical Society a remarkable series for $\sqrt{a^2+b}$, which had been discovered by C. A. Roberts, as follows:

$$\sqrt{a^2+b} = a + \frac{b}{2a}\,(1 - \frac{1}{q_1}\,(1 + \frac{1}{q_2}\,(1 + \frac{1}{q_3}\,(1 + \dots \tag{2}$$

where

$$q_1 = \frac{4a^2}{b} + 2\,;\; q_2 = q_1^2 - 2\,;\; q_3 = q_2^2 - 2\,\dots. \tag{3}$$

Since the q's increase to infinity, it is evident that this series converges for any value of a and b as I proved in my article "On the method of continued identity," Annals of Mathematics, Vol. 5, page 3, page 92. (This was an attempt to apply this form to the development of any function, which, however, was not practically successful except in yielding a remarkably convenient method of solving higher numerical equations.)

After that I published in the Mathematical Magazine "A new and expeditious method of computing the square root," which already contained the essential features of the general method though somewhat hidden. It is as follows:

We have identically

$$\sqrt{a^2+b} = \frac{b}{2a}\sqrt{\frac{4a^2}{b^2}(a^2+b)} = \frac{b}{2a}\sqrt{\tfrac{1}{4}q_1^2 - 1} \text{ if } q_1 = \frac{4a^2}{b} + 2$$

* Read before the Philosophical Society, Washington, D. C.

similarly

$$\sqrt{\tfrac{1}{4}\,q_1^2 - 1} = \frac{1}{q_1}\sqrt{q_1^2\,(\tfrac{1}{4}\,q_1^2 - 1)} = \frac{1}{q_1}\sqrt{\tfrac{1}{4}\,q_2^2 - 1} \text{ if } q_2 = q_1^2 - 2$$

$$\sqrt{\tfrac{1}{4}\,q_2^2 - 1} = \frac{1}{q_2}\sqrt{q_2^2\,(\tfrac{1}{4}\,q_2^2 - 1)} = \frac{1}{q_2}\sqrt{\tfrac{1}{4}\,q_3^2 - 1} \text{ if } q_3 = q_2^2 - 2\,.$$

We have then

$$\begin{aligned}\sqrt{a^2 + b} &= \frac{b}{2a}\sqrt{\tfrac{1}{4}\,q_1^2 - 1} = \frac{b}{2a}\cdot\frac{q_1}{2} \text{ approx.}\\ &= \frac{b}{2a}\cdot\frac{1}{q_1}\sqrt{\tfrac{1}{4}\,q_2^2 - 1} = \frac{b}{2a}\cdot\frac{1}{q_1}\cdot\frac{q_2}{2} \text{ approx.}\\ &= \frac{b}{2a}\cdot\frac{1}{q_1}\cdot\frac{1}{q_2}\sqrt{\tfrac{1}{4}\,q_3^2 - 1} = \frac{b}{2a}\cdot\frac{1}{q_1}\cdot\frac{1}{q_2}\cdot\frac{q_3}{2} \text{ approx.}\end{aligned} \tag{4}$$

etc.

These approximate values, since they are expressed by the identical auxiliaries q as in Robert's series, are likewise convergent for any a and b. They are indeed the sums of the terms of that series since

$$\begin{aligned}a + \frac{b}{2a} &= \frac{b}{2a}\cdot\frac{q_1}{2}\\ a + \frac{b}{2a}\left[1 - \frac{1}{q_1}\right] &= \frac{b}{2a}\cdot\frac{1}{q_1}\cdot\frac{q_2}{2}\\ a + \frac{b}{2a}\left[1 - \frac{1}{q_1}\left[1 + \frac{1}{q_2}\right]\right] &= \frac{b}{2a}\cdot\frac{1}{q_1}\cdot\frac{1}{q_2}\cdot\frac{q_3}{2},\end{aligned} \tag{5}$$

etc.

The generalization of this method for the nth root did not occur to me because then the successive remainders could not be $= 1$, and I considered this erroneously essential. The following, however, is the correct mode of generalization :

We have

$$(a^n \pm b)^{\frac{1}{n}} = \frac{1}{na^{n-1}}\,[(na^{n-1})^n\,(a^n \pm b)]^{\frac{1}{n}} = \frac{1}{na^{n-1}}\,(a_1^n - b_1)^{\frac{1}{n}} \tag{6_1}$$

if

$$a_1 = na^n \pm b\,;\; b_1 = a_1^n - (na^{n-1})^n\,(a^n \pm b) \tag{7_1}$$

$$(a_1^n - b_1)^{\frac{1}{n}} = \frac{1}{na_1^{n-1}}\,[(na_1^{n-1})^n\,(a_1^n - b_1)]^{\frac{1}{n}} = \frac{1}{na_1^{n-1}}\,(a_2^n - b_2)^{\frac{1}{n}} \tag{6_2}$$

if

$$a_2 = na_1^n - b_1\,;\; b_2 = a_2^n - (na_1^{n-1})^n\,(a_1^n - b_1)\,, \tag{7_2}$$

etc., and we have

$$(a^n \pm b)^{\frac{1}{n}} = \frac{1}{na^{n-1}}\cdot\frac{1}{na_1^{n-1}}\cdot\frac{1}{na_2^{n-1}}\cdots\frac{1}{na_{m-1}^{n-1}}\,(a_m^n - b_m)^{\frac{1}{n}}\,. \tag{6}$$

Now whatever the ratio between a^n and b, even if $b > a^n$ it follows from (7)

$$a_1{}^n > b_1\,;\ a_2{}^n > b_2, \text{ etc.},$$

and the binomial series can be applied without restriction to any form of (6), and if the remainders $b_1, b_2, \ldots$ are small enough with respect to $a_1{}^n, a_2{}^n, \ldots$ to be neglected, we have the approximate values:

$$(a^n \pm b)^{\frac{1}{n}} = \frac{1}{na^{n-1}} \cdot a_1, \text{ approximately,}$$

$$= \frac{1}{na^{n-1}} \cdot \frac{1}{na_1{}^{n-1}} \cdot a_2, \text{ approximately,} \tag{8}$$

etc.

In order to prove that these approximate values converged to the true value we give to the factorial (6) another form. Let

$$\frac{b}{a^n} = x\,;\ \frac{b_1}{a_1{}^n} = x_1\,;\ \frac{b_2}{a_2{}^n} = x_2, \text{ etc.} \tag{9}$$

Then we have:

$$(a^n \pm b)^{\frac{1}{n}} = a\,(1 \pm x)^{\frac{1}{n}} = \frac{1}{na^{n-1}} \cdot a_1\,(1 - x_1)^{\frac{1}{n}} = a\left[1 \pm \frac{x}{n}\right](1 - x_1)^{\frac{1}{n}} \tag{10_1}$$

$$= a\left[1 \pm \frac{x}{n}\right]\left[1 - \frac{x_1}{n}\right](1 - x_2)^{\frac{1}{n}} \tag{10_2}$$

. . . .

$$= a\left[1 \pm \frac{x}{n}\right]\left[1 - \frac{x_1}{n}\right]\left[1 - \frac{x_2}{n}\right]\cdots\left[1 - \frac{x_{m-1}}{n}\right](1 - x_m)^{\frac{1}{n}}, \tag{10_m}$$

and the scale of relation becomes:

$$x_1 = 1 - \left[1 \pm \frac{x}{n}\right]^{-n}(1 \pm x)\,, \tag{11_1}$$

$$x_2 = 1 - \left[1 - \frac{x_1}{n}\right]^{-n}(1 - x_1)\,, \tag{11_2}$$

. . . .

$$x_m = 1 - \left[1 - \frac{x_{m-1}}{n}\right]^{-n}(1 - x_{m-1})\,. \tag{11_m}$$

Since even if $x > 1$ we have necessarily $x_1\,;\ x_2, \ldots, x_m < 1$, we may apply

the binomial series to the second terms, except the first, if $\frac{x}{n} > 1$. We have then

$$x_m = 1 - \left[1 + x_{m-1} + \frac{n+1}{2n} x_{m-1}^2 + \frac{n+1}{2n} \cdot \frac{n+2}{3n} x_{m-1}^3 + \ldots\right](1 - x_{m-1})$$

$$= \left[1 - \frac{n+1}{2n}\right] x_{m-1}^2 + \frac{n+1}{2n}\left[1 - \frac{n+2}{3n}\right] x_{m-1}^3 + \ldots$$

$$= \frac{n-1}{2n} x_{m-1}^2 + \frac{n+1}{n} \cdot \frac{n-1}{3n} x_{m-1}^3 + \ldots . \qquad (12)$$

If, therefore, x_{m-1} is a small quantity of the first order then x_m is of the second order, and the smaller x_{m-1} the more nearly we can put

$$x_m = \frac{n-1}{2n} x_{m-1}^2 .$$

Since, then, the x diminish continually, therefore (10), and hence, also (8), are *always* convergent.

If n is not an integer, and therefore $\frac{1}{n}$ any proper or improper fraction, these conclusions are not affected and these factorials remain convergent though practically useless since then the scales of relation require extractions of high roots. The forms given are thus restricted in practice to the extraction of high roots, and if $N^{\frac{p}{q}}$ was required we place

$$N^{\frac{1}{q}} = (a^q \pm b)^{\frac{1}{q}} = a(1 \pm x)^{\frac{1}{q}}$$

and raise the result to the pth power.

In case $b > a^n$ or $x > 1$, the forms are however so slowly convergent that they are impracticable, while the binomial series is absurd.

Additional Note. Instead of deriving (10) from (6) it can be formed independently as follows:

We have $(1 + x)^{\frac{1}{n}} = 1 + \frac{x}{n}$, approximately, using two terms of the binomial series. Assume

$$(1 + x)^{\frac{1}{n}} = \left[1 + \frac{x}{n}\right](1 - x_1)^{\frac{1}{n}}$$

then

$$x_1 = 1 - \left[1 + \frac{x}{n}\right]^{-n}(1 + x), \text{ the same as } (11_1).$$

In the same manner we can take

$$(1+x)^{\frac{1}{n}} = \left[1 + \frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right](1-x_1)^{\frac{1}{n}} \tag{13}$$

whence the scale of relation

$$x_1 = 1 - \left[1 + \frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]^{-n}(1+x)\,. \tag{14$_1$}$$

But we have

$$\begin{aligned} x_1 = 1 - \Bigg[1 - \frac{n}{1}\left[\frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right] + \frac{n}{1} \cdot \frac{n+1}{2}\left[\frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]^2 \\ - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}\left[\frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]^3 + \ldots\Bigg](1+x) \\ = \left[\frac{n(n+1)}{1\,.\,2} \cdot \frac{n-1}{n^3} + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3n^3} - \frac{n(n+1)}{1\,.\,2n^2}\right]x^3 + \ldots \\ = \frac{(n+1)(n-1)}{1\,.\,2\,.\,3}\,\frac{x^3}{n^2} + \ldots, \end{aligned}$$

thus if x is small of the first order, x_1 is of the third, and the convergence of the factorial

$$\begin{aligned} (1+x)^{\frac{1}{n}} = \left[1 + \frac{x}{n} - \frac{n-1}{2} \cdot \frac{x^2}{n^2}\right]\left[1 - \frac{x_1}{n} - \frac{n-1}{2} \cdot \frac{x_1^2}{n^2}\right] \ldots \\ \left[1 - \frac{x_{m-1}}{n} - \frac{n-1}{2} \cdot \frac{x_{m-1}^2}{n^2}\right](1-x_m)^{\frac{1}{n}} \end{aligned}$$

is cubic.

By taking four terms of the binomial series for first factor we can form a factorial which has a quartic convergence and so on. However, since the scales of relation thus become more complicated, it is not apparent that there is any advantage in increasing the convergence in this manner. Moreover, if $x > 1$, then the first step becomes less advantageous the more terms are taken in the first factor.